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Kink Movements and Percolation in the Binary Additive Cellular Automaton

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We show that under the Bernoulli initial condition two kinks in the cellular automaton (CA) 18/256 will annihilate each other with probability one. It turns out that there is an equivalent statement in terms of percolation in the simple binary additive CA. Namely, under the Bernoulli initial condition, 1's do not percolate in the binary additive CA.

KEY WORDS: Cellular automata; random walk; percolation.

1. KINK MOVEMENTS IN THE CELLULAR AUTOMATON 18/256

In some cellular automata (CA) certain invariant configurations or phases can be distinguished. Consider, e.g., the elementary one-dimensional CA no. 18/256, which evolves according to the rules

$$001 \mapsto 1, \qquad 100 \mapsto 1$$

all the other triples $\mapsto 0$. Formally, we have a discrete-time dynamical system $(\eta_t;\, t=0, 1,...)$ with state space $\{0, 1\}^{\times \mathbb{Z}}$ and transition rule $\eta_{t+1} = T\eta_t$, where

$$(T\eta)(i) = 1 \qquad \text{if} \quad (\eta(i-1), \eta(i), \eta(i+1)) = (0, 0, 1) \text{ or } (1, 0, 0)$$
$$(T\eta)(i) = 0 \qquad \text{otherwise}$$

(For an account of cellular automata see, e.g., ref. 5.)

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It is easy to see that the configurations with zeros in every second site are invariant. In fact if we cancel these extra zeros, then the remaining system behaves according to the binary additive rule 6/16,

$$\alpha\beta \mapsto \alpha + \beta \pmod 2), \qquad \alpha, \beta \in \{0, 1\}$$

The CA 6/16 has state space $\{0, 1\}^{\times \mathbb{Z}} \cup \{0, 1\}^{\times(\mathbb{Z}+1/2)}$. Its transition rule T is

$$(T\eta)(i) = \eta(i - \tfrac{1}{2}) + \eta(i + \tfrac{1}{2}) \pmod 2)$$

By convention we assume that at time $t = 0$ the initial configuration $\in \{0, 1\}^{\times \mathbb{Z}}$. Hence at even time epochs the configurations $\in \{0, 1\}^{\times \mathbb{Z}}$, whereas at odd epochs they $\in \{0, 1\}^{\times(\mathbb{Z}+1/2)}$.

Suppose now that in the above kind of initial configuration for the CA 18/256, where we have zeros at every second site, we remove one of the zeros to get an initial configuration with two *phases* separated by a *kink*. A kink is by definition a block κ of the form $\kappa = 10^{2m}1$, where $2m$ (= the number of zeros between the ones at the ends) is even. We will call $2m$ the *length* of the kink and denote it by $|\kappa_0|$. If we look at the evolution of the CA 18 starting from an initial configuration $\eta = \eta_0$ with one kink $\kappa = \kappa_0$ we can easily see that this situation prevails, i.e., at every time $t = 0, 1, 2,...$ there exists exactly one kink κ_t in the configuration η_t. We will call κ_{t+1} the *successor* of the kink κ_t, $t = 0, 1,....$ Clearly, if $|\kappa_t| = 2m > 0$, then $|\kappa_{t+1}| = 2m - 2$ and the midpoints of κ_t and κ_{t+1} coincide. If $|\kappa_t| = 0$, i.e., κ_t is of the form 11, then the place and length of its successor κ_{t+1} depend also on the exterior of κ_t. By the transformation rules of 18/256 the right (resp. left) endpoint of κ_{t+1} is below the first block of the form 100 (resp. 001) to the right (resp. left) of κ_t. We illustrate the movement of the kink by the following picture (the movement of the midpoint of the kinks is illustrated by the continuous line):

```
η0   · · · 0 1 0 1 0 0 0 1 0 0|0 0 1 0 1 0 0 0 1 0 1 · · ·
η1   · · · · 0 0 0 1 0 1 0 1 0|0 1 0 0 0 1 0 1 0 0 · · · ·
η2   · · · · · 0 1 0 0 0 0 0 1|1 0 1 0 1 0 0 0 1 · · · · ·
η3   · · · · · · 0 1 0 0 0 1 0 0 0|0 0 0 1 0 1 · · · · · ·
η4   · · · · · · · 0 1 0 1 0 1 0 0|0 0 1 0 0 · · · · · · ·
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In ref. 1 it was proved that under the Bernoulli initial condition a single kink performs a random walk (with independent delay times). The Bernoulli initial condition (with one kink) means that in the initial configuration there exist two phases separated by one kink κ_0. At the free sites (i.e., at the sites where also 1's are allowed) outside κ_0 the 0's and 1's are distributed according to the Bernoulli (1/2, 1/2) distribution. This result confirmed the earlier simulation results obtained by Grassberger.[2]

Suppose now that we have exactly two kinks $\kappa_0^{(1)}$ and $\kappa_0^{(2)}$ in the initial configuration η_0. It follows from the local character of the transition rules that as long as $\kappa_0^{(1)}$ and $\kappa_0^{(2)}$ remain separate they move like single kinks, i.e., have their successor kinks $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$, $t = 1, 2,\ldots$ as described before. But now, if at some moment, say τ, both of them are of the form $\kappa_\tau^{(1)} = \kappa_\tau^{(2)} = 11$ and if between them the configuration is comprised of an alternating sequence of 0's and 1's, i.e., η_τ is, e.g., of the form

$$\eta_\tau = \cdots 1\ 0\ 0\ 0\ 1\ 0\ \overbrace{1\ 1}^{\kappa_\tau^{(1)}}\ 0\ 1\ 0\ 1\ 0\ \overbrace{1\ 1}^{\kappa_\tau^{(2)}}\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 1 \cdots \tag{1.1}$$

then at time $\tau + 1$ the kinks annihilate each other and so there are no kinks left in $\eta_{\tau+1}$. For example, if η_τ is as in (1.1), then

$$\eta_{\tau+1} = \cdots 1\ 0\ 1\ 0^{15}\ 1\ 0\ 0\ 0\ 1 \cdots$$

Simulation results[2] show that under the Bernoulli initial condition two kinks evolve like two independent RW's, implying in particular that annihilation ought to occur almost surely.

This, indeed, is the case:

Theorem 1. Under a Bernoulli initial condition the annihilation of two kinks occurs almost surely.

As in ref. 1, the proof is based on an isomorphism between the CA 18/256 and the binary additive CA 6/16 (see Theorem 2). In ref. 3 a related "linearization" of the CA 18/256 is constructed. The key lemma in the proof is a percolation result concerning the binary additive CA which may be of independent interest and is therefore stated as a separate theorem (Theorem 3).

2. KINK MOVEMENTS IN THE BINARY ADDITIVE CA

As noted in Section 1, the CA 18/256 having only one phase (i.e., no kinks) evolves on its free sites like the CA 6/16. (We call the sites where 1's are allowed *free*.)

Suppose now that an initial configuration η_0 for CA 18/256 has two phases separated by one kink

$$\kappa_0 = 10^{2m}1, \qquad m \geqslant 0$$

Let us map η_0 to an initial configuration $\tilde{\eta}_0$ for the CA 6/16 as follows:

Outside κ_0 we remove the nonfree sites (all of which are occupied by 0's). From κ_0 we remove m zeros and equip the midpoint with a bar to get a block $\tilde{\kappa}_0$ of the form

$$\tilde{\kappa}_0 = 10^k\underline{0}0^k1, \qquad \text{if} \quad m = 2k+1 \text{ is odd}$$

$$\tilde{\kappa}_0 = 10^k\_0^k1, \qquad \text{if} \quad m = 2k \text{ is even}$$

So the underscore indicating the midpoint is under the midzero if m is odd and between the two midzeros if m is even. We will call $\tilde{\kappa}_0$ still a *kink*.

Now we let the CA 6/16 evolve from $\tilde{\eta}_0$ according to its additive rule. Let us equip the midpoint of the unique block under $\tilde{\kappa}_0$ which is of the form 10^n1 again with an underscore and call it the *successor kink* $\tilde{\kappa}_1$ of $\tilde{\kappa}_0$. Similarly, $\tilde{\kappa}_1$ will have its successor kink $\tilde{\kappa}_2$, and so on. Thus, formally we have instead of the ordinary CA 6/16 *a marked* CA 6/16. It is evident that the (ordinary) CA 18/256 with one kink is isomorphic as a dynamical system to the marked CA 6/16 with one kink.

Now this isomorphism carries over to the case of an arbitrary number of kinks if we postulate that two kinks (i.e., blocks of type 10^m1 equipped with the underscores) annihilate each other (i.e., the underscore is removed) at the epoch of coalescence. In the case of coalescence of several kinks annihilation occurs if the number of the coalescing kinks is even, whereas one kink survives if the number is odd. We illustrate this by the following example of evolution of the marked CA 6/16 having initially two kinks:

$$\begin{array}{c}
\cdots 0\ 0\ 0\ 0\ 1_1\ 1\ 0\ 0\ 1\ 0_0\ 1\ 1\ 1\ 0\ 1\ 0 \cdots \\
\cdots 0\ 0\ 0\ 1\ 0_0\ 1\ 0\ 1\ 1\ \underline{0}\ 1\ 0\ 0\ 1\ 1\ 1 \cdots \\
\cdots 0\ 0\ 1\ 1\ \underline{0}\ 1\ 1\ 1\ 0\ 1_1\ 1\ 0\ 1\ 0\ 0 \cdots \\
\cdots 0\ 1\ 0\ 1_1\ 0\ 0\ 1\ 1\ 0_0\ 1\ 1\ 1\ 0 \cdots \\
\cdots 1\ 1\ 1\ \underline{0}\ 1\ 0\ 1\ 0\ 1\ \underline{0}\ 1\ 0\ 0\ 1 \cdots \\
\cdots 0\ 0\ 1_1\ 1\ 1\ 1\ 1\ 1_1\ 1\ 0\ 1 \cdots \\
\cdots 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1 \cdots \\
\cdots 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0 \cdots
\end{array} \tag{2.1}$$

We see from the picture that the two kinks get annihilated at $t = 7$.

Note that if in the picture, e.g., the 101 block to the right from the $1\underline{0}01$ kink were designated as a kink $1\underline{0}1$, then its successor would still be alive at $t = 7$. If the 1001 block in between the two kinks 1_1 and $1\underline{0}01$ were a kink $1\underline{0}01$, then it would coalesce at $t = 7$ with the two other kinks with the result that one kink would survive from this coalesence of an odd number of kinks.

To illustrate the isomorphism, we picture the isomorphic image of the example (2.1). The image follows the rules of the CA 18/256. We indicate the evolution of the midpoints of the two links again by lines. Note that contrary to the case of the marked CA 6/16, the lines do not belong to the model as necessary marks, but serve only for illustrational purposes:

$\cdots\ 0^5$ 0 0 0 0 1 | 1 0 1 0 0 0 0 0 1 0 0 | 0 0 1 0 1 0 1 0 0 0 1 0 0 0 $\cdots$
$\cdots\ 0^4$ 0 0 0 1 0 0 | 0 0 1 0 0 0 1 0 1 0 | 0 1 0 0 0 0 0 1 0 1 0 1 0 $\cdots$
$\cdots\ 0^3$ 0 0 1 0 1 0 | 0 1 0 1 0 1 0 0 0 1 | 1 0 1 0 0 0 1 0 0 0 0 0 $\cdots$
$\cdots\ 0^2$ 0 1 0 0 0 1 | 1 0 0 0 0 0 1 0 1 0 0 | 0 0 1 0 1 0 1 0 0 0 $\cdots$
$\cdots$ 0 1 0 1 0 1 0 | 0 1 0 0 0 1 0 0 0 1 0 | 0 1 0 0 0 0 0 1 0 $\cdots$
$\cdots$ 0 0 0 0 0 1 | 1 0 1 0 1 0 1 0 1 0 1 | 1 0 1 0 0 0 1 0 $\cdots$
$\cdots$ 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 $\cdots$
$\cdots$ 0 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 $\cdots$

If the $10^5 1$ block between the two kinks were instead a $10^4 1$ kink (i.e., the isomorphic image of a kink $1\underline{0}01$), then instead of the $10^{15}1$block we would have a $10^{14}1$ kink at time $t = 7$ [cf. the discussion after example (2.1)].

As a consequence of the above discussion, we see that we can as well study the annihilation problem within the framework of the simpler CA 6/16. So Theorem 1 is equivalent to the following result:

Theorem 2. Let us distribute the initial configuration for the CA 6/16 according to the Bernoulli (1/2, 1/2) distribution and let us designate any two $100\cdots01$ blocks as kinks. Then these two kinks will annihilate each other almost surely.

Let us record also the corollaries for the multikink case:

Corollary 1. Consider the CA 6/16 with Bernoulli initial condition and n kinks. Then eventually there are n (mod 2) kinks left (a.s.).

Corollary 2. Consider the CA 6/16 with Bernoulli initial condition and infinitely many kinks. Then eventually any of the kinks will be annihilated by some of the other kinks (a.s.).

Corollary 3. Consider the CA 18/256. Consider an initial condition obtained as follows. Let there be one phase, i.e., 1's are allowed only on every second site. Let 0's and 1's be distributed on these free sites with Bernoulli (1/2, 1/2) distribution. Now create an arbitrary number $2 \leqslant n \leqslant \infty$ of kinks by removing zeros from those 100...01 blocks which are wanted to become kinks. If n is finite, then there are eventually n (mod 2) kinks left (a.s.). If n is infinite, then eventually any of the kinks will be annihilated by some of the other kinks (a.s.).

3. PERCOLATION IN THE BINARY ADDITIVE CA

The proof of Theorem 2 will be based on the observation that annihilation of the two kinks is equivalent to *nonpercolation* of 1's. More precisely, let $\mathbb{N} = \{0, 1,...\}$, $\mathbb{Z} = \{..., -1, 0, 1, 2,...\}$, $\omega = (\omega(i);\ i \in \mathbb{Z})$, be an initial configuration for the CA 6/16. Let $\omega_t = (\omega_t(i))$ be the configuration at time $t \geqslant 0$.

A sequence $\sigma = (\sigma_0,..., \sigma_t)$ of sites is called a 1-*path* (from $(i, 0)$ to (j, t)) if:

(i) $\sigma_0 = i,\ \sigma_t = j$.

(ii) $\omega_s(\sigma_s) = 1$ for all $s = 0, 1,..., t$.

(iii) $|\sigma_s - \sigma_{s-1}| = 1/2$ for all $s = 1,..., t$.

t is called the *length* of σ.

Note that, if for any t and j, $\omega_t(j) = 1$, then there is a 1-path from some (i, 0) to (j, t). This follows from the fact the dynamics of 6/16 produces a 1 only from the two blocks 01 and 10 (both containing a 1).

Let a_t (resp. b_t) denote the right (resp. left) end site of the left kink (resp. right kink) at time t. Clearly any 1-path σ starting in between a_0 and b_0 stays between the two kinks (until possible annihilation), that is, if $a_0 \leqslant \sigma_0 \leqslant b_0$, then $a_t \leqslant \sigma_t \leqslant b_t$ for all $t \leqslant \tau$, where

$$\tau + 1 \doteq \text{the epoch of annihilation}$$

Also, since $\omega_{\tau+1}(i) = 0$ for $a_\tau - 1/2 \leqslant i \leqslant b_\tau + 1/2$, it follows that any 1-path starting between the kinks has length $\leqslant \tau$. On the other hand, since, e.g., $\omega_\tau(a_\tau) = 1$, and as remarked before there always exist unending 1-paths backwards, there always exist 1-paths starting from some $(i, 0)$ between the kinks and having length τ.

Thus we obtain the following characterization for $\tau \doteq$ the annihilation epoch -1:

$$\tau = \sup\{t \geqslant 0\text{: there is a 1-path of length } t \text{ from some } (i, 0) \text{ with } a_0 \leqslant i \leqslant b_0\}$$

Hence, in order to prove that annihilation occurs (with probability 1) we have to prove that (with probability one) there do not exist 1-paths of infinite length starting from any $(i, 0)$ between the kinks.

So we have still one equivalent formulation for Theorems 1 and 2:

Theorem 3. Let us distribute the initial configuration for the CA 6/16 according to the Bernoulli (1/2, 1/2) distribution. Then with probability one there are no infinite 1-paths.

Proof of Theorem 3. There is no loss of generality to consider paths starting from (0, 0). For any initial configuration $\omega = (\omega(i); i \in \mathbb{Z})$ let

$$\mathscr{S}(\omega) := \bigcup \{(j, t): \sigma_t = j \text{ for an infinite 1-path } \sigma \text{ starting from } (0, 0)\}$$

Let $A = \{\omega: \mathscr{S}(\omega) \neq \varnothing\}$. Let P denote the Bernoulli (1/2, 1/2) distribution of the initial configuration ω. First we claim that A is symmetric in the following sense:

$$\begin{aligned} &P\{(j-\tfrac{1}{2}, s+1) \in \mathscr{S} \mid A; \sigma_0, ..., \sigma_s; \sigma_s = j\} \\ &\quad = P\{(j+\tfrac{1}{2}, s+1) \in \mathscr{S} \mid A; \sigma_0, ..., \sigma_s; \sigma_s = j\} \end{aligned} \tag{3.1}$$

for all j and s, all paths $\{(\sigma_0, 0), (\sigma_1, 1), ..., (\sigma_s, s)\} \subset \mathscr{S}$.

In order to prove this, let, for a fixed $\omega \in A$, α denote the right-hand edge of the cone generated by the initial segment $(\sigma_0, ..., \sigma_s)$ of an infinite 1-path $\sigma = (\sigma_0, \sigma_1, ...)$ starting from (0, 0). Suppose $\sigma_s = j$. That means

$$\alpha = (\alpha_0, ..., \alpha_s) \doteq \left(\omega_0\left(j + \frac{s}{2}\right), \omega_1\left(j + \frac{s-1}{2}\right), ..., \omega_s(j)\right)$$

Similarly, let β denote the left-hand edge

$$\beta = (\beta_0, ..., \beta_s) \doteq \left(\omega_0\left(j - \frac{s}{2}\right), \omega_1\left(j - \frac{s-1}{2}\right), ..., \omega_s(j)\right)$$

and γ the upper edge

$$\gamma = (\gamma_0, ..., \gamma_s) \doteq \left(\omega_0\left(j - \frac{s}{2}\right), \omega_0\left(j - \frac{s}{2} + 1\right), ..., \omega_s\left(j + \frac{s}{2}\right)\right)$$

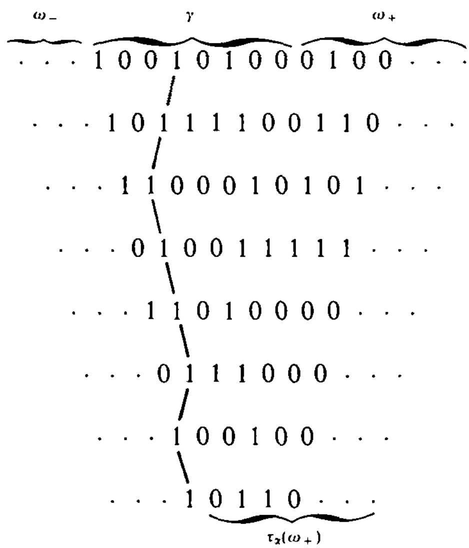
The configurations to the right and left from γ are denoted by

$$\omega_+ \doteq \left(\omega_0\left(j + \frac{s}{2} + 1\right), \omega_0\left(j + \frac{s}{2} + 2\right), ...\right)$$

$$\omega_- \doteq \left(\omega_0\left(j - \frac{s}{2} - 1\right), \omega_0\left(j - \frac{s}{2} - 2\right), ...\right)$$

respectively.

In what follows we take s and the 1-path $\sigma_0, ..., \sigma_s$ to be fixed. Hence so are the blocks α, β, and γ. Let τ_α be the map $\{0, 1\}^{\times \mathbb{N}} \to \{0, 1\}^{\times \mathbb{N}}$ which maps the sequence ω_+ to the sequence $(\omega_s(j+1), \omega_s(j+2), ...)$, where $(j, s) = (1/2, 7)$, $\alpha = 00111101$, $\beta = 11101011$, $\omega_+ = 0100\cdots$, and $\tau_\alpha(\omega_+) = 0110\cdots$):



(3.2)

Clearly τ_α is invertible and preserves the Bernoulli (1/2, 1/2) product measure.

Similarly, let τ_β be the map which maps the sequence ω_- to the sequence $(\omega_s(j-1), \omega_s(j-2),...)$. τ_β is invertible and measure-preserving, too.

Let the overbar denote the operation which turns any sequence $\omega = (\omega(n))$ to the sequence $\bar{\omega} = (\bar{\omega}(n))$, with $\bar{\omega}(-n) = \omega(n)$. Let B denote the event $B \doteq \{\omega\colon (-1/2, 1) \in \mathscr{S}(\omega)\}$. Note that

$$(j - \tfrac{1}{2}, s+1) \in \mathscr{S}(\omega) \tag{3.3}$$

if and only if

$$\overline{\tau_\beta(\omega_-)}\, 1 \tau_\alpha(\omega_+) \in B \tag{3.4}$$

Clearly, $(1/2, 1) \in \mathscr{S}(\omega)$ if and only if $\bar{\omega} \in B$. Hence

$$(j + \tfrac{1}{2}, s+1) \in \mathscr{S}(\omega) \tag{3.5}$$

if and only if

$$\overline{\tau_\alpha(\omega_+)}\, 1 \tau_\beta(\omega_-) \in B \tag{3.6}$$

Let φ be the map

$$\omega \mapsto \varphi(\omega) = \omega'$$

mapping a configuration $\omega = \bar{\omega}_- \gamma \omega_+$ (with a fixed γ) to a configuration $\omega' = \bar{\omega}'_- \gamma \omega'_+$ (of the same type), where

$$\tau_\beta \omega'_- = \tau_\alpha \omega_+, \qquad \text{i.e.,} \quad \omega'_- = \tau_\beta^{-1} \tau_\alpha \omega_+ \tag{3.7}$$

$$\tau_\alpha \omega'_+ = \tau_\beta \omega_-, \qquad \text{i.e.,} \quad \omega'_+ = \tau_\alpha^{-1} \tau_\beta \omega_- \tag{3.8}$$

Since τ_α and τ_β are invertible measure-preserving maps, it follows that so is φ. By (3.3)–(3.8)

$$\{(j - \tfrac{1}{2}, s+1) \in \mathscr{S}\} = \varphi^{-1}\{(j + \tfrac{1}{2}, s+1) \in \mathscr{S}\}$$

from which the assertion (3.1) follows.

Next we define a RW on A as follows:

Let $\omega \in A$ be fixed. First note that, since the Bernoulli product measure is a Lebesgue-type measure, there is a sequence $\xi^{(n)}$, $n = 0, 1,\dots$, of partitions of A, $\xi^{(n)} = \{A_1^{(n)}, A_2^{(n)},\dots, A_{2^n}^{(n)}\}$, such that

$$P(A_i^{(n)}) = 2^{-n} \qquad \text{and} \qquad A_{2i-1}^{(n+1)} \cup A_{2i}^{(n+1)} = A_i^{(n)}$$

for all i and n. We define inductively a sequence $\sigma_0, \sigma_1,\dots$ of $\mathbb{Z} \cup (\mathbb{Z} + 1/2)$-valued r.v.'s as follows:

Let $\sigma_0(\omega) = 0$, and inductively, assuming that $\sigma_0(\omega),\dots \sigma_{s-1}(\omega)$ and $\sigma_s(\omega) = j$ are given, we define:

(a) If $(j + \frac{1}{2}, s+1) \notin \mathscr{S}(\omega)$, then let $\sigma_{s+1}(\omega) = j - 1/2$.

(b) If $(j - 1/2, s+1) \notin \mathscr{S}(\omega)$, then let $\sigma_{s+1}(\omega) = j + 1/2$.

(c) If both $(j + 1/2, s+1)$ and $(j - 1/2, s+1) \in \mathscr{S}(\omega)$, then let $\sigma_{s+1}(\omega) = j \pm 1/2$ according as $\omega \in A_i^{(s)}$ for an even or odd i.

By (3.1) and by the construction,

$$P\{\sigma_{s+1} = j + \tfrac{1}{2} \mid A; \sigma_0,\dots, \sigma_{s-1}; \sigma_s = j\} = \tfrac{1}{2}$$

$$P\{\sigma_{s+1} = j - \tfrac{1}{2} \mid A; \sigma_0,\dots, \sigma_{s-1}; \sigma_s = j\} = \tfrac{1}{2}$$

In other words, $(\sigma_s; s = 0, 1,\dots)$ is a symmetric RW on A. Let Q be the probability measure P conditioned with respect to A: $Q(\cdot) = P(\cdot \mid A)$. It will turn out that Q is in fact singular with regard to P, implying $P(A) = 0$, as asserted by our theorem. The proof of this is based on the following picture:

Start a symmetric RW from the origin. Let us label its path by 1's. On the other hand, we think that these 1's are generated by the CA 6/16 with initial distribution Q. Let us follow the RW up to site (j, s):

```
1   0   0   1   0   1   0   0
           /
  1   0   1   1   1   1   0
         /
    1   1   0   0   0   1
         \
      0   1   0   0   1
           \                          (3.9)
        1   1   0   1
             \
          0   1   1
             /
            1   0
             \
              1
```

Here $(j, s) = (1/2, 7)$.

This path determines uniquely a block $\gamma = (\omega(j - s/2),..., \omega(j + s/2))$ of the initial configuration ω. Let $n \doteq j + s/2$ [in (3.9), $n = 4$]. Let us again denote by α the right edge of the cone generated by the RW $(\sigma_0,..., \sigma_s)$ [in (3.9), $\alpha = 00111101$]. Now, it follows from the transition rules of 6/16 that the symbol $\omega_{n+1} = \omega_{j+s/2+1}$ is determined by the parity of the occurrence of the next move to the right after s by the RW. More precisely, let

$$\nu \doteq \min\{n \geqslant 0: \sigma_{s+n+1} = \sigma_{s+n} + \tfrac{1}{2}\}$$

Now all the events $\{\nu = 0\}$, $\{\nu = 2\}$, $\{\nu = 4\}$,..., will produce the same symbol $\omega_{n+1} = f_n(\omega_n, \omega_{n-1},...)$, whereas the events $\{\nu = 1\}$, $\{\nu = 3\}$,..., produce $\omega_{n+1} = 1 - f_n(\omega_n, \omega_{n-1},...)$. Here f_n is a function of $\omega_n, \omega_{n+1},...$, depending only on finitely many of the variables $\omega_n, \omega_{n-1},....$ [In (3.9), $f_n(\omega_n, \omega_{n+1},...) = f_4(0, 0, 1, 0, 1, 0, 0, 1) = 0$, since a 0 at $(5, 0)$ is compatible with having ν even.] But this implies

$$Q\{\omega_{n+1} = f_n(\omega_n, \omega_{n-1},...) \mid \omega_n, \omega_{n-1},...\} = P\{\nu \text{ is even}\} = \tfrac{2}{3}$$
$$Q\{\omega_{n+1} = 1 - f_n(\omega_n, \omega_{n-1},...) \mid \omega_n, \omega_{n-1},...\} = P\{\nu \text{ is odd}\} = \tfrac{1}{3}$$

rendering Q singular with respect to the symmetric Bernoulli product measure P.

Concluding Remarks. The above proof proves only the a.s. annihilation of two kinks. If we would like to know, e.g., the exact distribution of the annihilation epoch, then by the percolation equivalence we ought to calculate the distribution of the maximum length of the 1-paths starting between the two kinks. (An alternative way to calculate this distribution and to prove the a.s. annihilation might be to consider the two random walks as some kind of Markovian stationary RWs.)

More generally, as noted by Grassberger,[2] the case where there is an infinite number of kinks becomes still more complicated and interesting. There one can ask what is the decay rate of the density of the kinks. Simulation results[2] show that the rate is $t^{-1/2}$, which is the rate in the case of ordinary i.i.d. RWs.

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